DETECTING DE-LAMINATIONS IN COMPOSITE BEAMS USING NATURAL FREQUENCIES AND THE BAYESIAN INFERENCE

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We show a mathematical model of the vibration of a laminated beam, which is made up of two elastic beams bonded together by a resilient layer. De-laminations are assumed to occur in this resilient layer. As the size and location of the de-lamination varies, the natural frequencies of the beam change. We shows how to use this variation and apply the Bayesian inference to de-lamination de-laminations. The Bayes' theorem tells us the relationship between the structural parameters and the natural frequencies with conditional distribution of those quantities. The desired distribution is numerically simulated using the Markov-Chain-Monte-Carlo (MCMC) algorithm. We further extend the method when the prior knowledge of an exact number of de-laminations is unknown.

1. Introduction

We present a method of detecting one or two de-laminations in a laminated beam and a t-beam (beam-reinforced plate) (see Figs. 1 and 2). First, the natural frequencies of the beams are calculated using the Fourier series solution of the elastic beams coupled by resilient bonding layer. Then, the Bayesian inference and the Markov-Chain-Monte-Carlo (MCMC) is used with the natural frequencies consisting of Gaussian error to estimate the locations and sizes of the de-lamination. The modelling technique here is similar to that of [1, 2, 3, 4], in which the natural frequencies are used to determine the presence of de-lamination. In these articles, it has been shown that small de-lamination affects the natural frequencies more than measurement errors. The accuracy of the various modelling methods varies and often cannot be estimated, because the number of specimens in reality is limited. In this paper, we instead study how to estimate the de-laminations with an assumption that the model can predict the natural frequencies accurately.

A linear system is constructed using the potential energy of the vibration of the beam/plate, including the energy contribution from the resilient boding layer. The de-lamination is represented as a gap in the bonding payer. We use the theory of elasticity that was founded on the Hamiltonian mechanics (chapter 2 of [8]). Formulating the linear system using the system of partial differential equations of the deflection of the beam may be more common. But using the integral forms of the Lagrangian of the system is mathematically equivalent. We have used the variational principle on the

Lagrangian because of its simplicity when incorporating the contribution from the bonding layer. The eigenvalues of the linear system are then computed, and thus the natural frequencies of the structures are found. The first 5 frequencies for the laminated beam and 10 for the t-beam are used for the MCMC. Then the Reversible jump(RJ)-MCMC [5] is used to estimate the unknown number of de-laminations. The results are represented as probability distribution curves of the locations and sizes of the de-laminations. We also present the distribution curves for the number of de-laminations when RJ-MCMC is used.

2. Method of solution

2.1 Forward problem

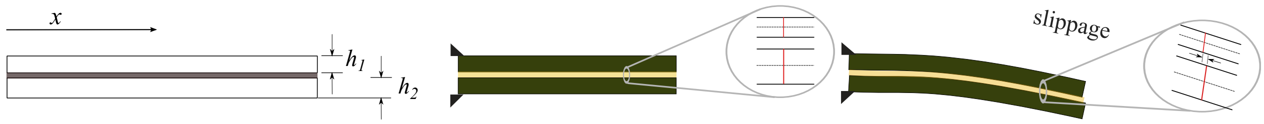

Figure 1: Schematic drawings of a clamped beam and its bending motion.

We first present a model of a laminated beam shown in Fig. 1. The assumptions for the beam are that its cross section is small compare to its length, and thus the Euler beam can be used to model its vibrations in the low to mid frequency range. The idea of using a resilient layer as a bonding between two beams has been used in [6] [7], which are listed int he references. The slippage model in this paper is an attempt to express the deformation of two completely slippery beams and completely rigidly bonded beams. We assume that a laminated beam may have a property somewhere between those extreme cases. There are studies using either of those cases, which have been given the references. The limit of our model is the same limit of the Euler beam model, that is, the shear deformation is omitted and the mid-plane does not stretch in the in-plane direction. As a consequence, our beam would not bend if the slippage constant were set very high, which is not true in real life. Hence, a laminated beam with very rigid bonding or a large shear deformation will have to modelled with a different regime.

The beam is clamped at one end. The bending motion of this Euler beams will be represented using the following eigenfunctions of the motion.

$$\psi_m(x) = \sqrt{\frac{1}{A}}\left[\cosh k_m x - \cos k_m x - \gamma_m \left(\sinh k_m x - \sin k_m x\right)\right], \quad m = 1, 2, ... \tag{1}$$

where $\gamma_m = (\cos k_m A + \cosh k_m A) / (\sin k_m A - \sinh k_m A)$ and the wavenumbers k_m are the solutions of $\cos k_m A \cosh k_m A = -1$, A being the length of the beam. The vertical displacement of the beam can be expressed as

$$w(x) = \sum_{m=1}^{N} C_m \psi_m(x), \tag{2}$$

where C_m are the coefficients yet to be determined. The roots of this equation are found numerically, though $k_m \approx \pi(2m-1)/2A$ for $m \geq 5$. Note that $\gamma_m \approx 1$ for $m \geq 5$.

The second case of the forward problem is a t-beam shown in Fig. 2. The clamped edge again breaks the symmetry of the position of the de-laminations. Only the vertical bending motion will be considered here, and thus the torsional and the in-plane deflections will not be included in the model.

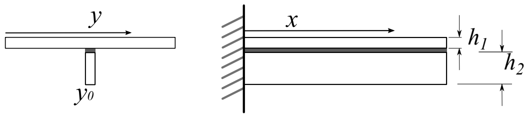

Figure 2: Axis orientation of the t-beam (not to scale) with one end clamped at $x = 0$.

Then the vertical displacement of the plate is given by

(3) $$w(x, y) = \sum_{m=1}^{M} \sum_{n=1}^{N} C_{mn} \psi_m(x) \phi_n(y),$$

where ϕ_n is the eigenfunctions for the y-direction when the edges $y = 0, B$ are free. The eigenfunctions are given by

$$\phi_n(y) = \sqrt{\frac{1}{B}} \left[\sinh \kappa_n y + \sin \kappa_n y + \gamma_n' \left(\cosh \kappa_n y + \cos \kappa_n y\right)\right], \quad n = 1, 2, ...$$

where $\gamma_n' = (\sin \kappa_n B - \sinh \kappa_n B) / (\cosh \kappa_n B - \cos \kappa_n B)$ the wavenumbers are the solutions of $\cos \kappa_n B \cosh \kappa_n B = 1$, B being the width of the top plate. Note that the series in Eq. (3) is truncated for later numerical computation. The parameter values used for the computation are, size (A, B) are (1.5 m, 0.3 m), mass density m is 500 kgm^{-3}, Young's modulus is E is 14 GPa, thickness is h_1 and 0.02 m (top plate), 0.05 m (beam), thickness of the laminate beam h_0, h_2 are 0.05 m, 0.1 m, Poisson ratio ν is 0.4 and the slippage constant s is 10^8 Nm^{-1}.

We now formulate the linear system using the variational method of the elastic beams and plates. The strain energy for a beam ([8]) is

(4) $$U_{\mathsf{b}} = \frac{EI}{2} \int_0^A (w''(x))^2 \, dx$$

where E is Young's modulus and I is moment of inertia, which is calculated by $I = h_0 h_1^3/12$, h_0 being the width of the beam. The strain energy of the composite beam will be sum of the two beams.

The strain energy of a plate is

(5) $$U_{\mathsf{p}} = \frac{D}{2} \int_0^A \int_0^B \left\{ (\nabla^2 w)^2 + 2(1 - \nu) \left[w_{xy}^2 - w_{xx} w_{yy} \right] \right\} \, dx dy$$

where the bending stiffness D is calculated by $D = Eh_1^3/12(1 - \nu^2)$ for the thickness h_1 and Poisson ratio ν. The strain energy of the reinforcement beams are given by

(6) $$U_{\mathsf{rb}} = \frac{EI}{2} \int_0^A (w_{xx}(x, y_0))^2 \, dx$$

where $y = y_0 = 0.15$ is the position of the reinforcement beam.

The additional energy due to the slippage (see Fig. 1) in the resilient layer [9] is

(7) $$U_{\mathsf{s}} = \frac{s}{2} \int_{\mathcal{D}} \left((h_1 w_x(x, y_0) + h_2 w_x(x, y_0)\right)^2 \, dx$$

where s denotes the slippage constant. The de-laminations are defined as $[c_j - l_j, c_j + l_j]$, $j = 1, 2, ..., N_{\mathsf{d}}$ where c_j is the centre of the de-lamination, $2l_i$ is the width and N_{d} is the number of de-laminations. $\mathcal{D}$ is the integral limit which has the de-lamination regions subtracted.

The linear systems are derived for the column vectors $\mathbf{C} = (C_1, C_2, ..., C_M)^{\mathrm{t}}$ for a beam and $\mathbf{C} = (C_{11}, C_{12}, ..., C_{MN})^{\mathrm{t}}$ for a plate. Then, the eigenvalue problem is given by

$$\mathcal{M}^{-1}\mathcal{U}\mathbf{C} - (2\pi f)^2\mathbf{C} = 0 \tag{8}$$

where $\mathcal{M}$ and $\mathcal{U}$ are the mass density and stiffness matrices, and $(2\pi f)^2$ is the eigenvalue, in other words f is the natural frequency.

The matrix for the eigenvalue problem can be constructed using Eqs. (4) to (7). Hence, the matrix for the beam case is

$$\mathcal{U}_{\mathsf{b}} = (EI_1 + EI_2)\left[k_1^4, k_2^4, ..., k_M^4\right] \tag{9}$$

where $[\cdots]$ is a diagonal matrix of k_m^4, and I_1 and I_2 are moment of inertia for the top and bottom beams, respectively.

The stiffness matrix for the plate with the reinforcement beams is

$$\mathcal{U}_{\mathsf{p}} = D\left[(k_1^2+\kappa_1^2)^2, (k_2^2+\kappa_1^2)^2, ..., (k_M^2+\kappa_N^2)^2\right] + EI\mathcal{T}^{\mathsf{t}}\left[k_1^4, k_2^4, ..., k_M^4\right]\mathcal{T} \tag{10}$$

where the matrix $\mathcal{T}$ represents the linear operation on the vector $\mathbf{c}$, and the m'th element of the vector $\mathcal{T}\mathbf{C}$ is given by

$$[\mathcal{T}\mathbf{C}]_m = \sum_{n=1}^{N} C_{mn}\phi_n(y_j)$$

The stiffness matrix $\mathcal{U}$ for the beam or the plate in Eq. (8) can be derived by summing all the energy matrices,

$$\mathcal{U} = \mathcal{U}_{\mathsf{b}} + \mathcal{U}_{\mathsf{s}} \quad \text{or for the plate} \quad \mathcal{U} = \mathcal{U}_{\mathsf{p}} + \mathcal{U}_{\mathsf{s}}$$

The mass density matrix of the laminated beam is diagonal. For the t-beam the matrix $\mathcal{M}$ is given by $\mathcal{M}_{\mathsf{p}} + \mathcal{T}^{\mathsf{t}}\mathcal{M}_{\mathsf{b}}\mathcal{T}$, where $\mathcal{M}_{\mathsf{b}}$ and $\mathcal{M}_{\mathsf{p}}$ are diagonal matrices of the mass density of the beams and the top plate, respectively.

2.2 Inverse problem

We test two algorithms, one with an assumption that the number of de-laminations is known, and the other without that assumption. When all de-laminations, $\mathbf{c} = \{c_j\}_{j=1}^{N_{\mathsf{d}}}$ and $\mathbf{l} = \{l_j\}_{j=1}^{N_{\mathsf{d}}}$ are known, the first N_{f} exact natural frequencies, denoted by $\{\mu_1, \cdots, \mu_{N_{\mathsf{f}}}\}$, can be computed using the forward map described in section 2.1. The inverse problem is a procedure that finds the de-laminations from the evaluates natural frequencies, which consist of random noise. The noise-free natural frequency is denoted by μ_i and is evaluated by solving the forward map. We assume that the noise, n_i, is normally distributed with the mean of μ_i and the standard deviation of $0.01\mu_i$ (1% of μ_i); $n_i|\mu_i \sim N(0, 0.01\mu_i)$. This noise could be a measurement error or a background noise. The set of natural frequencies $\{f_i\}_{i=1}^{N_f}$ used for the MCMC simulation are evaluated by adding the noise-free frequency and the error,

$$f_i = \mu_i + n_i\,, \qquad i = 1, \cdots, N_f\,.$$

Here, n_i is a random sample generated from $N(0, 0.01\mu_i)$.

We use the Bayesian inference to estimate the distribution for $\mathbf{c}$ and $\mathbf{l}$ from an observed set of natural frequencies $\mathbf{f} = \{f_i\}_{i=1}^{N_{\mathsf{f}}}$. This is called the posterior distribution and denoted by $p(\mathbf{c}, \mathbf{l}|\mathbf{f})$. If probability for $\mathbf{f}$ is non-zero, the posterior is represented by the Bayes' rule, that is

$$p(\mathbf{c}, \mathbf{l}|\mathbf{f}) = \frac{p(\mathbf{f}|\mathbf{c}, \mathbf{l})p(\mathbf{c}, \mathbf{l})}{p(\mathbf{f})} \propto p(\mathbf{f}|\mathbf{c}, \mathbf{l})p(\mathbf{c}, \mathbf{l})\,. \tag{11}$$

where $p(\mathbf{f}|\mathbf{c}, \mathbf{l})$ is the likelihood and $p(\mathbf{c}, \mathbf{l})$ is the prior. The posterior distribution is proportional to the likelihood and the prior.

Because of the assumption that a Gaussian random noise is in the natural frequency observation, the i-th observation f_i is normally distributed with the mean of μ_i and standard deviation of σ_i, the likelihood on the right hand side of Eq. (11) can be expressed by

$$p(\mathbf{f}|\mathbf{c},\mathbf{l}) = \prod_{i=1}^{N_\mathsf{f}} \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-(f_i-\mu_i)^2/2\sigma_i^2} \ . \tag{12}$$

The likelihood in Eq. (11) is assessed by a probability of $\mathbf{f}$ conditioned on using the exact natural frequency, μ, in which is computed for a given de-lamination

We assume that all de-laminations are within the beam with the length of A and neighbouring de-laminations do not overlap. The domain for de-lamination is $(c_j, l_j) \in [0, A] \times [0, A/2] \backslash Overlap$. The prior is formulated as the following.

$$p(\mathbf{c},\mathbf{l}) = \mathbb{1}(N_\mathrm{O} = 0) \prod_{j=1}^{N_\mathsf{d}} p(c_j)p(l_j) = \mathbb{1}(N_\mathrm{O} = 0) \prod_{j=1}^{N_\mathsf{d}} U(c_j|0, A)U(l_j|0, A/2) \tag{13}$$

where N_O is the number of overlapping de-laminations. If none of the de-laminations overlap, $\mathbb{1}(N_\mathrm{O} = 0) = 1$, otherwise $\mathbb{1}(N_\mathrm{O} = 0) = 0$. The uniform distribution, U, is used for both $p(c)$ and $p(l)$ in Eq. (13). Although a simple uniform prior is used here, a more informative prior could be considered for more complex problems.

The posterior given by Eq. (11) is found using the likelihood and the prior in Eq. (12) and Eq. (13), respectively. The analytical formula for the posterior is not available for this problem and thus the MCMC method is used to simulated the posterior numerically. A sequence of random samples from the posterior is simulated using the Metropolis Hastings algorithm and those samples contain the Markov chain property.

The Markov chain with the length of T is denoted by $\{\mathbf{c}^{(t)}, \mathbf{l}^{(t)}\}_{t=1}^{T}$. At each iteration, a sample $(\mathbf{c}, \mathbf{l})$ is generated by sampling (c_j, l_j) for each de-lamination in turn. At t-th iteration, with a given $\{\mathbf{c}^{(t-1)}, \mathbf{l}^{(t-1)}\}$, a sample $\{\mathbf{c}^{(t)}, \mathbf{l}^{(t)}\}$ is generated by the following algorithm.

Step 1 Set $j = 1$.
Step 2 Generate the proposals, $c^* \sim q(\cdot|c_j^{(t-1)})$ and $l^* \sim q(\cdot|l_j^{(t-1)})$.
Step 3 Set the vectors $\mathbf{c}^0 = \mathbf{c}' = \left[c_1^{(t)}, \cdots, c_{j-1}^{(t)}, c_j^{(t-1)}, \cdots, c_{N_\mathsf{d}}^{(t-1)}\right]$ and $\mathbf{l}^0 = \mathbf{l}' = \left[l_1^{(t)}, \cdots, l_{j-1}^{(t)}, l_j^{(t-1)}, \cdots, l_{N_\mathsf{d}}^{(t-1)}\right]$. Note that $\mathbf{c}^0 = \mathbf{c}' = \left[c_1^{(t-1)}, \cdots, c_{N_\mathsf{d}}^{(t-1)}\right]$ when $j = 1$, and similarly for $\mathbf{l}^0$. Replace the j-th element of $\mathbf{c}'$ and $\mathbf{l}'$ with $c'_j = c^*$ and $l'_j = l^*$, respectively.
Step 4 Compute the prior $p(\mathbf{c}', \mathbf{l}')$ in Eq. (13) and likelihood $p(\mathbf{f}|\mathbf{c}', \mathbf{l}')$ in Eq. (12) for $\mathbf{c}'$ and $\mathbf{l}'$.
Step 5 Compute the acceptance probability, $\alpha(c^*, l^*|c_j^{(t-1)}, l_j^{(t-1)})$ by

$$\alpha(c^*, l^*|c_j^{(t-1)}, l_j^{(t-1)}) = \min\left(1, \frac{p(\mathbf{f}|\mathbf{c}', \mathbf{l}')p(\mathbf{c}', \mathbf{l}')q(c_j^{(t-1)}|c^*)q(l_j^{(t-1)}|l^*)}{p(\mathbf{f}|\mathbf{c}^0, \mathbf{l}^0)p(\mathbf{c}^0, \mathbf{l}^0)q(c^*|c_j^{(t-1)})q(l^*|l_j^{(t-1)})}\right) .$$

Step 6 Accept c^* and l^* with $\alpha(c^*, l^*|c_j^{(t-1)}, l_j^{(t-1)})$. If they are accepted, set $c_j^{(t)} = c^*$ and $l_j^{(t)} = l^*$. Otherwise $c_j^{(t)} = c_j^{(t-1)}$ and $l_j^{(t)} = l_j^{(t-1)}$.
Step 7 Set $j = j + 1$ and repeat the Steps 2 – 6 until $j = N_\mathsf{d}$.

The proposal density q is the normal distribution with a mean of the condition and variance of φ^2, that is, $q(c^*|c_j^{(t-1)}) = N(c^*|c_j^{(t-1)}, \varphi_c^2)$ and $q(l^*|l_j^{(t-1)}) = N(l^*|l_j^{(t-1)}, \varphi_l^2)$ for $j = 1, \cdots, N_\mathsf{d}$. The variances φ_c^2 and φ_l^2 are related to the random walk sizes. Here the number of unknown parameters is small and φ_c^2 and φ_l^2 are manually adjusted following [10]. The adaptive MCMC [11] can be easily adapted for a larger number of unknown parameters [12].

We now assume that the number of de-laminations is unknown (but limited to 3) and the length of de-laminations is the same. The reversible jump (RJ) MCMC by [5] has been proposed to simulate a Markov chain for the posterior distribution on spaces of varying dimensions. A number of de-laminations, $N_{\min} \leq N_{\mathsf{d}} \leq N_{\max}$, is equally distributed

$$p(N_{\mathsf{d}} = N_{\min}) = \cdots = p(N_{\mathsf{d}} = N_{\max})$$

and the lengths of de-laminations are equal, i.e., $l = l_1 = \cdots = l_{N_{\mathsf{d}}}$. The probability of N_{d} becomes

$$p(N_{\mathsf{d}}|\mathbf{f}) = \frac{p(N_{\mathsf{d}}) \displaystyle\int \pi(\mathbf{f}_{obs}|\mathbf{c}_{N_{\mathsf{d}}}, l, N_{\mathsf{d}})\pi(\mathbf{c}_{N_{\mathsf{d}}}, l)\mathrm{d}(c_1, \cdots, c_{N_{\mathsf{d}}})\mathrm{d}l}{\displaystyle\sum_{N'_{\mathsf{d}}=N_{\min}}^{N_{\max}} p(N'_{\mathsf{d}}) \int \pi(\mathbf{f}_{obs}|\mathbf{c}_{N'_{\mathsf{d}}}, l, N'_{\mathsf{d}})\pi(\mathbf{c}_{N'_{\mathsf{d}}}, l)\mathrm{d}(c_1, \cdots, c_{N'_{\mathsf{d}}})\mathrm{d}l}. \tag{14}$$

The dimension of the parameter space varies with N_{d}. Two types of moves are proposed; two de-laminations are merged and a de-lamination is split to two. When a split move is proposed, a move from a lower dimensional space $[\mathbf{c}_{N_{\mathsf{d}}}, l, u]$ to a higher dimensional space $[\mathbf{c}'_{N_{\mathsf{d}}+1}, l']$ is determined by an invertible deterministic function with a continuous random variable u independent of $\mathbf{c}_{N_{\mathsf{d}}}$. The reverse of the move (merging de-laminations) can be completed by using the inverse transformation, so that the proposal is deterministic. Then the acceptance probability reduces to

$$\alpha((\mathbf{c}'_{N_{\mathsf{d}}+1}, l')|(\mathbf{c}_{N_{\mathsf{d}}}, l)) = \min\left\{1, \frac{p(\mathbf{c}'_{N_{\mathsf{d}}+1}, l'|\mathbf{f})r_{\text{merge}}(\mathbf{c}'_{N_{\mathsf{d}}+1}, l')}{p(\mathbf{c}_{N_{\mathsf{d}}}, l|\mathbf{f})r_{\text{split}}(\mathbf{c}_{N_{\mathsf{d}}}, l)q(u)}\left|\frac{\partial(\mathbf{c}'_{N_{\mathsf{d}}+1}, l')}{\partial(\mathbf{c}_{N_{\mathsf{d}}}, l, u)}\right|\right\} \tag{15}$$

where $r_{\text{split}}(\mathbf{c}_{N_{\mathsf{d}}}, l)$ is the probability of choosing a split move when in state $(\mathbf{c}_{N_{\mathsf{d}}}, l)$ and $q(u)$ is the density function of u. Note that the final term in the ratio above is a Jacobian arising from the change of variable from $(\mathbf{c}_{N_{\mathsf{d}}}, l, u)$ to $(\mathbf{c}'_{N_{\mathsf{d}}+1}, l)$. In this paper, the probabilities of the moves are equally weighted, $r_{\text{split}} = r_{\text{merge}} = 0.5$.

Split proposal A randomly chosen (c^*, l^*) is split into (c_1^*, l) and (c_2^*, l), with a random variable u from the beta distribution; $u \sim \text{Beta}(2, 2)$, and $c_1^* = c^* - ul^*$, $c_2^* = c^* + ul^*$ and $l = (1 - u^2)l^*$. The absolute determinant of Jacobian matrix is

$$\left|\frac{\partial(\mathbf{c}'_{N_{\mathsf{d}}+1}, l')}{\partial(\mathbf{c}_{N_{\mathsf{d}}}, l, u)}\right| = 2l^* + 2u^2 l^*$$

and the proposal is accepted using Eq. (15).

Merging proposal Randomly chosen two adjacent de-laminations (c_1^*, l) and (c_2^*, l) where $c_1^* < c_2^*$ are merged to one de-lamination (c^*, l^*) using the inverse of the above deterministic functions. The acceptance probability is an inverse of the fraction in Eq. (15).

Figures 3 and 4 show the probability distribution functions of the locations and the sizes of the de-laminations for a laminated beam and a t-beam. The distribution of the number of de-lamination is also shown with the RJ-MCMC results. We had to use a lower relative error of 0.5% for the t-beam to obtain meaningful results. We used a mid-range desktop PC. The regular MCMC took approximately 1 hour and the RJ-MCMC did approximately 2 hours using Matlab running on a single core at 3.4 GHz to complete 10^5 iterations.

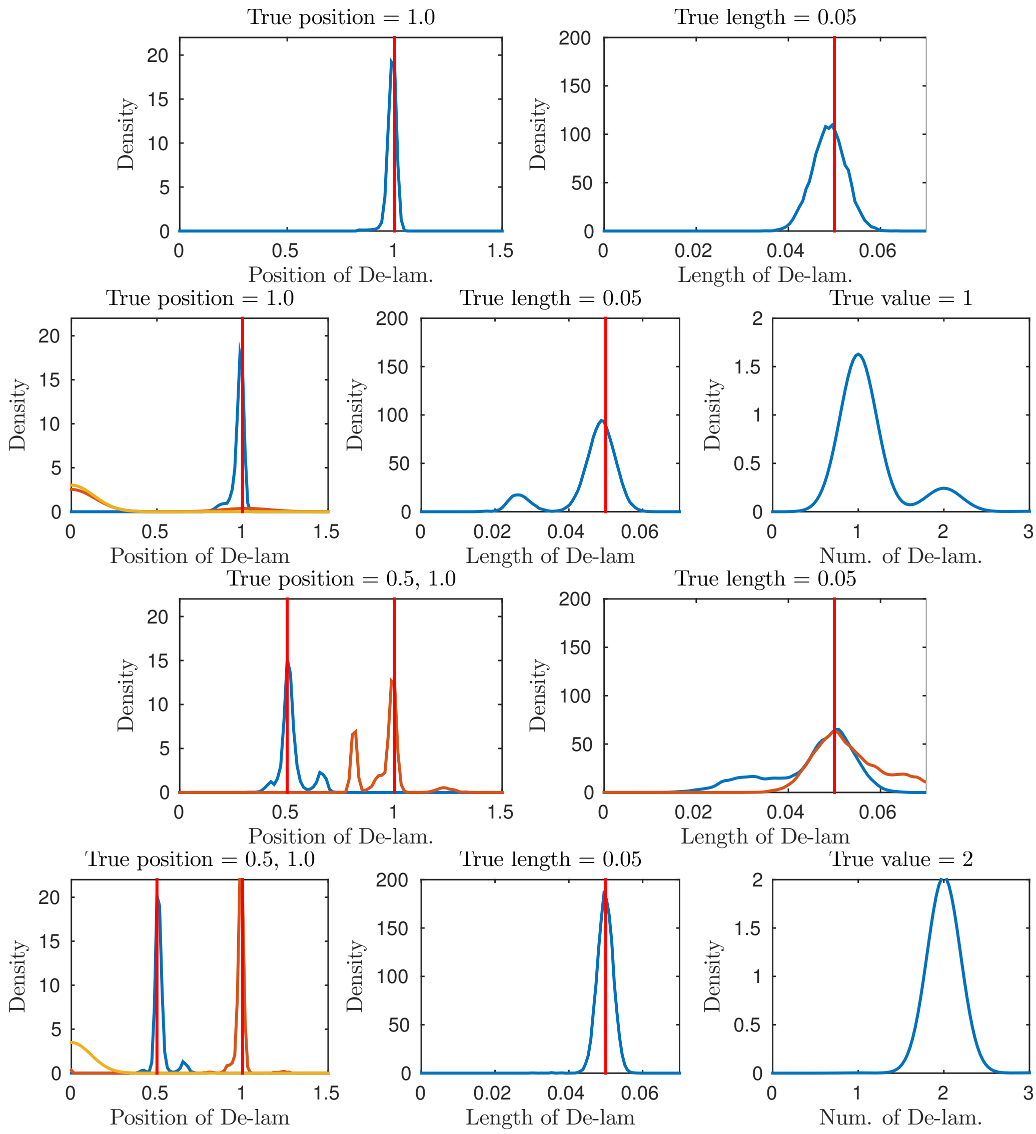

Figure 3: Marginal posterior distribution for the position and the length parameters for the 1 (top 2) and 2 (bottom 2) delaminations in a laminated beam. The number of de-laminations is known (top and 3rd rows) and unknown (2nd and 4th rows).

3. Summary

We have shown that the natural frequencies of a laminated beam and a t-beam can be used with the Bayesian inference to estimate the locations and sizes of de-laminations. It has not been validated by experiments how accurate the likelihood $p(\mathbf{f}|\mathbf{c}, \mathbf{l})$ is. The literature on this topic suggests that the existing models may be good enough for the MCMC method shown here. Assuming that the model is accurate, the MCMC and RJ-MCMC are able to estimate the de-lamination with 95% accuracy. An advantage of this method over the damage-index methods is that the we can give probability distribution of the de-lamination. A disadvantage is that this MCMC iterations take time. It has taken 10^5 iterations to obtain the results shown in the figures.

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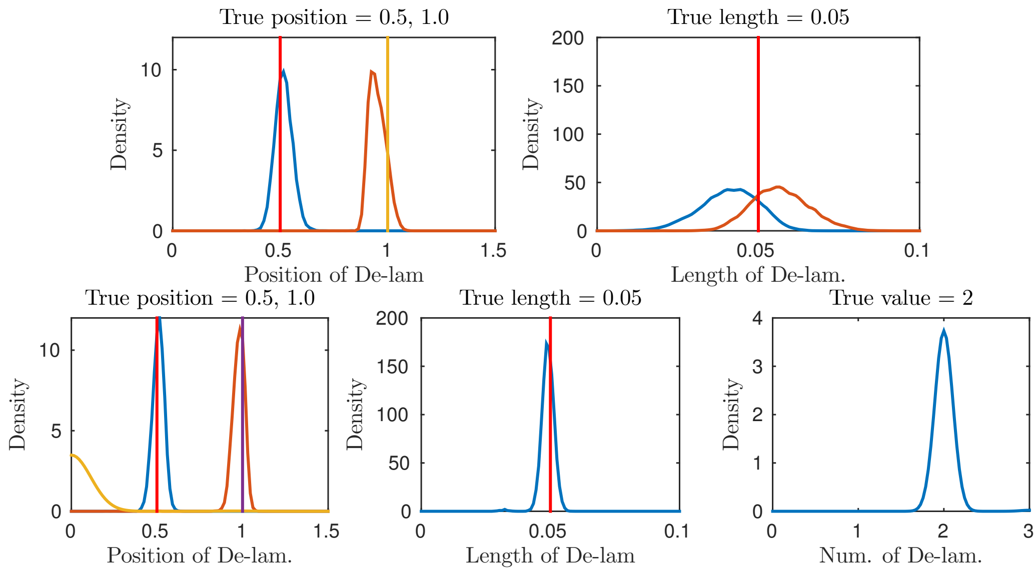

Figure 4: The same quantities as Fig 3 for a t-beam. The number of de-laminations is known (top) and unknown (bottom).

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