Detecting de-laminations in composite beams using natural frequencies and the Bayesian inference

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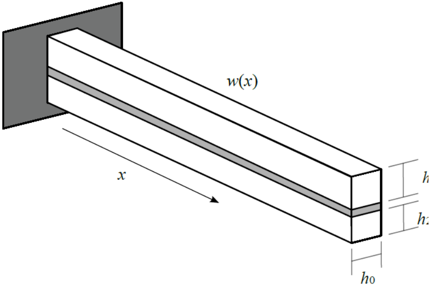
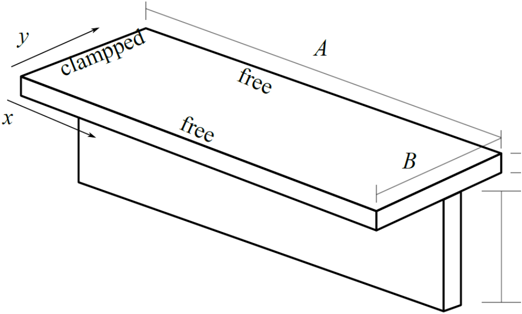
Introduction

Vibration of laminated beams: Two elastic beams glued together.





Objectives

Part 1

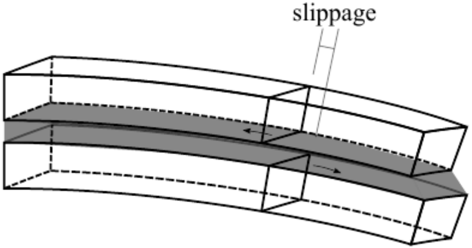
Find whether 10 natural frequencies are enough for the MCMC to estimate the de-lamination.

Part 2

Find whether the Reversible-jump MCMC can be used to estimate the number of de-laminations.

Natural frequencies of the beams

The linear bending motion of the beam with a thin bonding layer, i.e., there is some movement at the bonding layer.



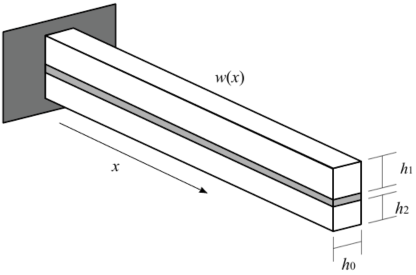
Error and simulation of de-laminations

Simulate the natural frequencies of the beam and add Gaussian noise to simulate the measurement error and inaccuracy in the model.

Bayes' theorem and MCMC

Use the noisy data to estimate the statistical properties (mean, variance, pdf) of the modelling parameters – location, size and number of de-laminations.

Laminated beam

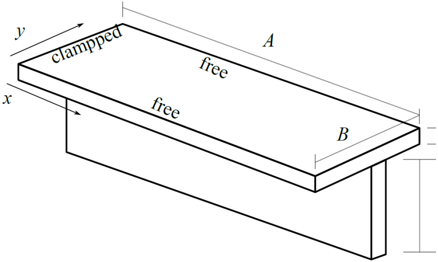


Vertical deflection of an Euler beam

$$w(x) = \sum_{m=1}^{N} C_m \psi_m(x)$$

$\psi_m(x)$: Eigenfunctions of one-end clamped beam.

T-Beam



A thin plate and a beam

$$w(x,y) = \sum_{m=1}^{N} C_{mn}\psi_m(x)\phi_n(y)$$

$\phi_n(y)$: Eigenfunctions of free-free beam

Forward Model

Eigenvalue problem

The linear systems are derived for the column vectors
$\mathbf{C} = (C_1, C_2, ..., C_M)^{\mathrm{t}}$ for a beam and
$\mathbf{C} = (C_{11}, C_{12}, ..., C_{MN})^{\mathrm{t}}$ for a plate.
Then, the eigenvalue problem is

$$\mathcal{U}\mathbf{C} - (2\pi f)^2 \mathcal{M}\mathbf{C} = 0$$

$\mathcal{M}$: Mass density
$\mathcal{U}$: Stiffness matrices
f: Natural frequency

Top beam strain energy

$$\frac{EI_1}{2}\int_0^A (w_1''(x))^2\,dx$$

Slippage energy

$$\frac{s}{2}\int_{\mathcal{D}} ((h_1 w_1'(x) + h_2 w_2'(x))^2\ dx$$

Bottom beam strain energy

$$\frac{EI_2}{2}\int_0^A (w_2''(x))^2\,dx$$

Total potential energy $\mathcal{U}$

Total kinetic energy $(2\pi f)^2\mathcal{M}$

Linear system

$$\mathbf{C}^t\left[\mathcal{U} - (2\pi f)^2\mathcal{M}\right]\mathbf{C}$$

Normal equation - Eigenvalue problem

$$\mathcal{U}\mathbf{C} \ - \ (2\pi f)^2\mathcal{M}\mathbf{C} \ = \ 0$$

Inverse Model

Bayes' theorem

$$p(\mathbf{c},\mathbf{l}|\mathbf{f}) = \frac{p(\mathbf{f}|\mathbf{c},\mathbf{l})p(\mathbf{c},\mathbf{l})}{p(\mathbf{f})} \propto p(\mathbf{f}|\mathbf{c},\mathbf{l})p(\mathbf{c},\mathbf{l})$$

where $p(\mathbf{f}|\mathbf{c},\mathbf{l})$ is the likelihood and $p(\mathbf{c},\mathbf{l})$ is the prior. $\mathbf{c}$ and $\mathbf{l}$ from an observed set of natural frequencies $\mathbf{f} = \{f_i\}_{i=1}^{N_f}$. This is called the posterior distribution and denoted by $p(\mathbf{c},\mathbf{l}|\mathbf{f})$.

Stochastic representation of vibration

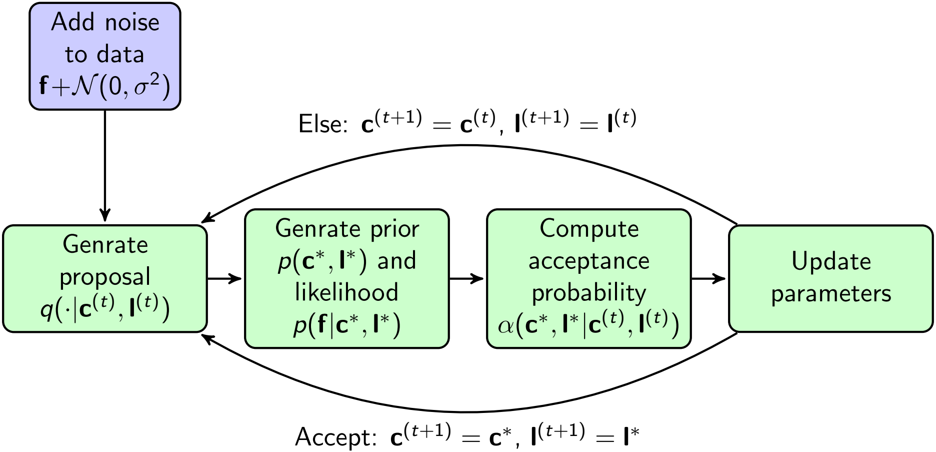
$$p(\mathbf{c}, \mathbf{l}|\mathbf{f}) \propto p(\mathbf{f}|\mathbf{c}, \mathbf{l})p(\mathbf{c}, \mathbf{l})$$

$p(\mathbf{f}|\mathbf{c}, \mathbf{l})$: Likelihood of getting the frequencies $\mathbf{f}$ from a set of parameters $(\mathbf{c}, \mathbf{l})$ $\Leftrightarrow$ Forward problem

$p(\mathbf{c}, \mathbf{l})$: Prior distribution $\Leftrightarrow$ Restrictions on de-laminations, e.g., no overlap, within the beam, limited number.

$p(\mathbf{c}, \mathbf{l}|\mathbf{f})$: Distribution of $(\mathbf{c}, \mathbf{l})$ from observed frequencies $\mathbf{f}$ $\Leftrightarrow$ Inverse problem

MCMC simulation



Forward solver

The bending motion of a cantilever beam can be represented using the eigenfunctions:

$$\psi_m(x) = \sqrt{\frac{1}{A}}\left[\cosh k_m x - \cos k_m x - \gamma_m \left(\sinh k_m x - \sin k_m x\right)\right]$$

$$\gamma_m = \left(\cos k_m A + \cosh k_m A\right) / \left(\sin k_m A - \sinh k_m A\right)$$

k_m: Roots of $\cos k_m A \cosh k_m A = -1$. They are found numerically, though $k_m \approx \pi(2m-1)/2A$ for $m \geq 5$, and $\gamma_m \approx 1$ for $m \geq 5$.

MCMC with Reversible Jump

Regular MCMC

Create a chain with acceptance probability for $\alpha(c^*, l^*|c_j^{(t-1)}, l_j^{(t-1)})$ by

$$\alpha(c^*, l^*|c_j^{(t-1)}, l_j^{(t-1)}) = \min\left(1, \frac{p(\mathbf{f}|\mathbf{c}', \mathbf{l}')p(\mathbf{c}', \mathbf{l}')q(c_j^{(t-1)}|c^*)q(l_j^{(t-1)}|l^*)}{p(\mathbf{f}|\mathbf{c}^0, \mathbf{l}^0)p(\mathbf{c}^0, \mathbf{l}^0)q(c^*|c_j^{(t-1)})q(l^*|l_j^{(t-1)})}\right)$$

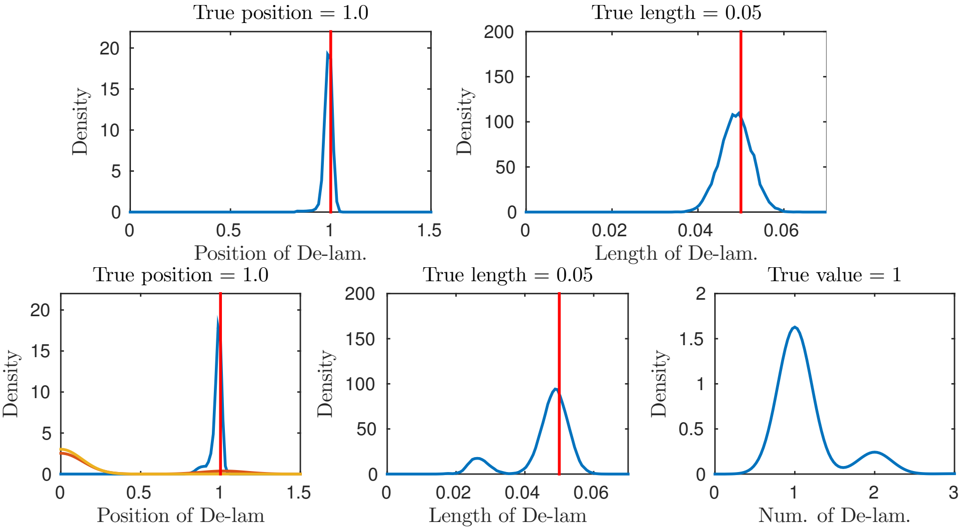
$p(\mathbf{f}|\mathbf{c}, \mathbf{l})$'s are computed by solving the eigenvalue problem.

Number of parameters unknown

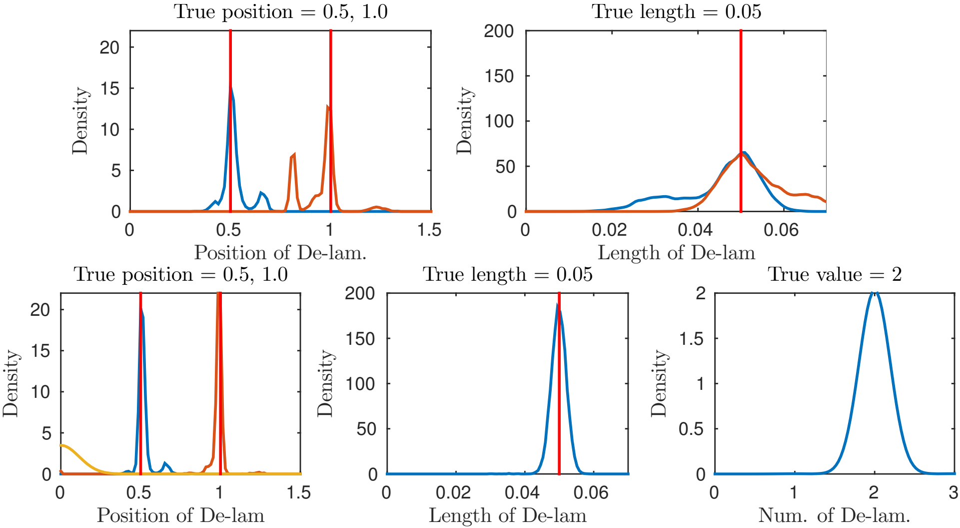
Split proposal A randomly chosen (c^*, l^*) is split into (c_1^*, l) and (c_2^*, l), with a random variable u from the beta distribution; $u \sim \text{Beta}(2,2)$, and $c_1^* = c^* - ul^*$, $c_2^* = c^* + ul^*$ and $l = (1 - u^2)l^*$.

Merging proposal Randomly chosen two adjacent de-laminations (c_1^*, l) and (c_2^*, l) where $c_1^* < c_2^*$ are merged to one de-lamination (c^*, l^*).

MCMC simulation - 1 de-lamination



MCMC simulation - 2 de-laminations



Summary

1. Fast forward problem solver: Linear system - eigenvalue problem
2. Inverse problem with unknown number of de-laminations - Reversible-jump MCMC
3. RJ-MCMC takes long chains to converge $\approx 10^5$ iterations and 2 hours of computing time.