FREE-BODY AND FLEXURAL MOTION OF A FLOATING ELASTIC PLATE IN PROXIMITY TO A WALL

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Abstract

A model of a floating elastic plate needs to include the draft of the plate in order to compute the surge motion. One difficulty with such model is that Green's function for the boundary value problem has varying orders of singularities on the boundary and at the corners. This paper shows how to resolve the boundary integration over the corners around the fluid domain necessary in order for accurate representation of water-tank experiments. The singularities are separated using the Kummer transform, and then the associated boundary integrals are evaluated analytically. Hence the numerical computation avoids the singular parts altogether. The zero-thickness approximation is often used to describe the simple harmonic flexural motions of the plate, because one can use modal expansions of the velocity potential, which is computationally straightforward. However, such approximation does not allow the surge motion, which is always present and particularly important in water-tank scaled experiments.

Introduction

Motion of an one dimensional floating body has 3 degrees of freedom, surge, heave, and rotational (ref: Newman). When the floe is elastic, the flexural motions includes heave and rotation, which can be described by the theories of incompressible fluid and thin elastic plate (ref). The surge motion needs to taken into account when the size of the floe is comparable to the wavelength of the motion, which is the case for most wave-tank experiments and off-shore structures. The theoretical model then has to have the draft (submerged part) of the plate. Therefore the often used model, floating thin plate model, in which the thickness of the plate is effectively ignored, discards the surge motion.

In this paper, the boundary integral method (BIM) is used to compute the all three degrees of motions of the plate for realistic water-tank experiment situations, which include the wave-maker at one end, length and thickness of the plate (see figure). Water-tank experiments usually have the floe fixed by some form of elastic constraints, such as a vertical rod or mooring to the floor of the tank. Therefore one needs to anticipate the appropriate amount of constraints given by the spring constant and damping in such rod or mooring systems (ref). The BIM lets us deal with the boundary of the fluid domain, denoted by Ω_w, which has corners and a moving wall. Our innovation in this method is to derive numerically stable BIEs, in which the singularities of the Green's function at the corners of the boundary. The singularities are separated using the Kummer transform. Then the boundary integrals containing those singular parts are evaluated analytically, consequently the numerical computation involves no singular parts.

The resulting BIEs are solved using the Galerkin method based on the eigenfunctions of a elastic beam and the orthogonal set of Gegenbauer polynomials. The leading terms of the displacement solution represents the heave and roll. The edge conditions for the plate may be varied by using the associated eigenfunctions without changing the computational procedure. The following sections will a brief instruction of derivation of the BIEs for the body of water with simple harmonic waves from a wave maker. Then the solutions will be shown in the forms of velocity potential and surge motion by arbitrarily accurate series expansions over orthogonal polynomials.

Governing Equations

The two-dimensional geometry is defined by the horizontal and vertical Cartesian coordinates, x and z respectively. In the absence of the plate, the fluid occupies the domain $\Omega = \{x, z : x > 0, -h < z < 0\}$, so that $z = 0$ defines the equilibrium position of the fluid surface, $z = -h$ is the floor of the tank and $x = 0$ is the location of the wave maker.

Under the regular assumptions of linear motions, the fluid's velocity field is sought as the gradient of a velocity potential $\Phi = \Phi(x, z, t)$. Imposing time-harmonic conditions the velocity potential is defined as $\Phi(x, z, t) = \Re\mathrm{e}\{(g/\mathrm{i}\omega)\phi(x, z)\mathrm{e}^{-\mathrm{i}\omega t}\}$, where $g \approx 9.81\,\mathrm{m\,s^{-2}}$ is acceleration due to gravity, ω is a prescribed angular momentum and ϕ is a (reduced) potential that must be calculated.

The potential ϕ satisfies Laplace's equation throughout the fluid domain, that is $\nabla^2\phi = 0$ for $(x, z) \in \Omega$. On the floor of the tank, $z = -h$, the no-flow condition $\partial_z\phi = 0$ is imposed. At the linearized free-surface, $z = 0$, the condition $\partial_z\phi = \sigma\phi$ holds, where $\sigma = \omega^2/g$ is a frequency parameter. Waves are generated through a prescribed horizontal velocity $\partial_x \phi = v(z)$ at the linearized position of the wave maker $x = 0$. The potential must also describe outgoing waves in the far field $x \to \infty$.

A thin-elastic plate occupies the interval $a < x < b$, where $0 < a < b$ and $b - a \equiv l$ is the length of the plate. In equilibrium the lower surface of the plate is located at $z = -d = -\rho_p D/\rho_w$, where D is the thickness of the plate, $\rho_p =$ is its density, and $\rho_w =$ is the fluid density. The linearized fluid domain in the presence of the plate is therefore $\Omega_w \equiv \Omega \backslash \{x, z : a < x < b, -d < z < 0\}$.

Fluid motion causes the plate to oscillate, and the position of its lower surface at time t is denoted $z = -d + \Re\mathrm{e}\{\xi(x)\mathrm{e}^{-\mathrm{i}\omega t}\}$. The displacement function ξ is related to the potential ϕ through the linearized equations

$$\phi = F\xi'''' + (1 - \sigma d)\xi, \quad \partial_z \phi = \sigma \xi, \tag{1a}$$

where $F = F_0/(\rho_w g)$, and $F_0 \propto D^3$ is the flexural rigidity of the plate.

The plate is also permitted to surge to and fro, although these motions are restrained by a mooring system. The horizontal position of the plate, $\Re\mathrm{e}\{u\mathrm{e}^{-\mathrm{i}\omega t}\}$ say, is coupled to the potential ϕ by the linearized equation of motion

$$(S - \sigma M - \mathrm{i}A)u = \rho_w \int_{-d}^{0} \{\phi(b, z) - \phi(a, z)\}\, \mathrm{d}z, \tag{1b}$$

where $M = \rho_p D l$ is the mass of the plate, S in the spring constant, and $A = \omega A_0/g$, in which A_0 is the damping constant. The constant u and the potential ϕ are also coupled by the kinematic conditions

$$\partial_x \phi(a, z) = \partial_x \phi(b, z) = \sigma u \quad (-d < z < 0). \tag{1c}$$

Solution Method

Consider the Green's function $G = G(x, z|x_0, z_0)$, which is defined as the solution to the above problem in the absence of the plate and wavemaker, and for which wave motion is forced at the source point (x_0, z_0). That is, it satisfies

$$\nabla^2 G = \delta(x - x_0)\delta(z - z_0) \quad (x, z) \in \Omega,$$

$\partial_z G = 0$ on $z = -h$, $\partial_z G = \sigma G$ on $z = 0$, $\partial_x G = 0$ on $x = 0$, and G represents outgoing waves as $x \to \infty$. The function may be calculated in a straightforward manner to be

$$G = \frac{1}{2\mathrm{i}} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{\mathrm{i}k_n(x+x_0)} + \mathrm{e}^{\mathrm{i}k_n|x-x_0|}}{k_n c_n} w_n(z) w_n(z_0),$$

where k_n $(n \in \mathbb{N})$ are the roots k of the dispersion relation $k \tanh(kh) = \sigma$, the vertical functions $w_n(z) = \cosh\{k_n(z + h)\}$, and the constants $c_n = ||w_n||^2$.

Applying Green's theorem in the plane to ϕ and G over Ω_p produces the integral expression

$$\epsilon\phi = f - \int_\Gamma \{(\partial_{n_0} G)\phi_0 - G(\partial_{n_0}\phi_0)\}\, \mathrm{d}s_0, \tag{2}$$

where a subscript 0 indicates that a function is evaluated at (x_0, z_0) rather than (x, z). The integral is around the wetted surface of the plate Γ, with tangential coordinate s and (outward) normal n. The forcing term f contains the contribution of the wave maker. The quantity ϵ is defined as $\epsilon_0/2\pi$, where ϵ_0 is the angle around the point (x, z) in Ω_p.

A system of integral equations are formed from (2) by allowing the field point (x, z) tend to the three continuous components of Γ, in turn. For what follows, these components will be denoted $\Gamma_{a\,(b)} = \{x, z : x = a\,(b), -d < z < 0\}$ and $\Gamma_d = \{x, z : a < x < b, z = -d\}$.

Singularities and Corners

It is well established that a Green's function in a two-dimensional plane contains a logarithmic singularity at the point at which the field and source points coincide, i.e. $(x, z) = (x_0, z_0)$. In the definition of G, the singularity is manifest as the non-convergence of the series at this point. The presence of the singularity will clearly impede the numerical evaluation of the integrals in which it appears.

A standard method for dealing with this issue is to apply a Kummer transformation. This involves the subtraction of the asymptotic limit of the terms of the series representation of G, which is balanced by the addition of this second series in closed form. The logarithmic singularity can then be further manipulated into a more convenient form. The result is that the Green's function may be expressed as

$$G = \widetilde{G} + \frac{1}{2\pi}\Big(\log(\mathcal{R}_-) + \log(\mathcal{R}_+)\Big),$$

where $\widetilde{G}$ is a bounded function and

$$(x - x_0) + \mathrm{i}(z \pm z_0) = \mathcal{R}_\pm \mathrm{e}^{\mathrm{i}\Theta_\pm}.$$

The singularity is contained solely in the term $\log(\mathcal{R}_-)$. However, the term $\log(\mathcal{R}_+)$ is also separated due to its near-singular for small values of the draft d.

The method for dealing with integrals involving G is demonstrated with the following example. Consider

$$\int_a^b G(\partial_{n_0}\phi_0)\,\mathrm{d}x_0 = \int_a^b \widetilde{G}(\partial_{n_0}\phi_0)\,\mathrm{d}x_0 + \frac{1}{2\pi}\sum_{i=\pm}\int_a^b \log(\mathcal{R}_i)(\partial_{n_0}\phi_0)\,\mathrm{d}x_0.$$

The integrals involving the logarithmic functions are then written

$$\int_a^b \log(\mathcal{R}_\pm)(\partial_{n_0}\phi_0)\,\mathrm{d}x_0 = \partial_n\phi\int_a^b \log(\mathcal{R}_\pm)\,\mathrm{d}x_0 + \int_a^b \log(\mathcal{R}_\pm)(\partial_{n_0}\phi_0 - \partial_n\phi)\,\mathrm{d}x_0.$$

The first integral on the right hand side of the above equation does not involve an unknown function and can be calculated explicitly. The function beneath the second integral tends to zero as $x \to x_0$ and can therefore be evaluated numerically at a low cost.

It is often stated that as a Green's function traverses a contour its normal derivative is bounded. However, this is not the case for contours that contain corners, such as Γ. Specifically, a first-order singularity occurs when the field and source points tend to the same corner from opposing limits. An approach is described here for circumventing this issue.

The approach is demonstrated through another example. Consider

$$\int_a^b (\partial_{n_0}G)\phi_0\,\mathrm{d}x_0 = \int_a^b (\partial_{n_0}\widetilde{G})\phi_0\,\mathrm{d}x_0 + \frac{1}{2\pi}\sum_{i=\pm}\int_a^b (\partial_{n_0}\log(\mathcal{R}_i))\phi_0\,\mathrm{d}x_0.$$

The singular terms are then extricated by writing

$$\int_a^b (\partial_{n_0}\log(\mathcal{R}_\pm))\phi_0\,\mathrm{d}x_0 = \phi\int_a^b (\partial_{n_0}\log(\mathcal{R}_\pm))\,\mathrm{d}x_0 + \int_a^b (\partial_{n_0}\log(\mathcal{R}_\pm))(\phi_0 - \phi)\,\mathrm{d}x_0.$$

The function beneath the final integral is bounded as $x \to x_0$ and can therefore be evaluated numerically. The integrals of the normal derivative of the logarithmic function alone can be treated analytically by noting the following identity

$$\partial_{n_0}\log(\mathcal{R}_\pm) = -\partial_{s_0}\Theta_\pm,$$

(see **?**). Accounting for the jump in Θ_- at the point $x = x_0$, it can be shown that

$$\int_a^b (\partial_{n_0}\log(\mathcal{R}_-))\,\mathrm{d}x_0 = 0,$$

whereas Θ_+ is continuous along the interval and therefore

$$\int_a^b (\partial_{n_0}\log(\mathcal{R}_+))\,\mathrm{d}x_0 = -\Big[\Theta_+\Big]_{x_0=a}^b \quad (z = z_0 = -d).$$

Expansions

Applying the above modifications, the system of integral equations are now open to numerical solution. This is achieved using a form of the Galerkin technique, in which the unknown functions are approximated as a linear combination of a finite set of chosen trial functions. Inner-products of the integral equations are then taken in turn with the members of corresponding set of test functions.

First though, the number of unknowns present in the integral equations is reduced by implementing the boundary conditions (1a,c). This leaves the system of integral equations to be solved for the constant u and the functions $\phi_i = \phi_i(z) \equiv \phi(i,z)$ $(-d < z < 0)$ for $i = a, b$, and $\xi = \xi(x)$.

Let the displacement function be approximated by

$$\xi(x) \approx \frac{2}{l}\sum_{m=0}^{M}\xi_m X_m(\hat{x}); \quad \hat{x} = \frac{2}{l}(x-a) - 1,$$

for some chosen constant M. The orthonormal set $\{X_m\}$ are the eigenfunctions of the spectral problem

$$X_m'''' - \alpha_m^4 X_m = 0 \quad (-1 < \hat{x} < 1),$$

with boundary conditions $X'' = X''' = 0$ $(\hat{x} = -1, 1)$, and corresponding eigenvalues α_m. Note that the boundary conditions are the natural conditions for a plate with free ends. The above expansion is particularly convenient for analysis of the motion of the plate, as the first mode X_0 supports its heave motion and the second mode X_1 supports its roll motion. The subsequent modes X_m $(m \geq 2)$ describe the flexural motion of the plate.

The potential functions at the ends of the plate are similarly approximated, with

$$\phi_i(z) \approx c_i + \frac{2}{d}\sum_{n=0}^{N}\phi_{i,n}\mathcal{C}_{2n}(\hat{z}); \quad \hat{z} = \frac{z}{d}, \qquad (3)$$

for $i = a, b$ and for some chosen constant N. The functions $\mathcal{C}_{2n}$ are a set of weighted even Gegenbauer polynomials (see page 561 of **?**[1]). The weighting is chosen so

[1] $C_n^{(\alpha)}(z) = \sum_{k=0}^{[n/2]}(-1)^k \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)k!(n-2k)!}(2z)^{n-2k}$

that the expected singular behaviour of the fluid velocity at the submerged corners of the plate are captured in the approximation (see **?**).

Inner-products of the integral equations on the ends of the plate $\Gamma_{a(b)}$ are taken with the set of unweighted even Gegenbauer polynomials, and the set $\{X_m\}$ is used as the test functions for the integral equation posed on Γ_d. Extra equations must also be added in order to close the system. Firstly, the condition (1b) is applied in conjunction with the approximations (3). Continuity of the potential is also be ensured at the submerged corners of the plate, which sets the values of the constants c_i $(i = a, b)$. These continuities are also important from a numerical standpoint when evaluating certain integrals using the method outlined in the previous section.

Numerical Results

References